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"State-Space Interpretation of
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Abstract

A model predictive control technique based on a step response model is developed using state estimation techniques. The standard step response model is extended so that integrating systems can be treated within the same framework. Based on the modified step response model, it is shown how the state estimation techniques from stochastic optimal control can be used to construct the optimal prediction vector without introducing significant additional numerical complexity. In the case of integrated or double integrated white noise disturbances filtered through general first-order dynamics and white measurement noise, the optimal filter gain is parametrized explicitly

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in terms of a single parameter between 0 and 1, thus removing the requirement for solving a Riccati equation and equipping the control system with useful on-line tuning parameters. Parallels are drawn to the existing MPC techniques such as Dynamic Matrix Control (DMC), Internal Model Control (IMC) and Generalized Predictive Control (GPC).

1 Introduction

Model Predictive Control (MPC) has emerged as a powerful practical control technique during the last decade. Its strength lies in that it uses step (or impulse) response data which are physically intuitive, and that it can handle hard constraints explicitly through on-line optimization. Various MPC techniques such as Dynamic Matrix Control (DMC) (Cutler & Ramaker, 1980), Model Algorithmic Control (MAC) (Rouhani & Mehra, 1982), and Internal Model Control (IMC) (Garcia & Morari, 1982) have demonstrated their effectiveness in industrial applications during the past 10 years (Richalet, *et al.*, 1978; Cutler & Ramaker, 1980; Cutler & Hawkins, 1988). One drawback of these "traditional" MPC techniques has been that, because they are developed in an unconventional manner using step response models, their generalization to more complex cases has been difficult. For example, most of the traditional techniques incorporate feedback into the algorithm in an *ad hoc* way, such as by adding a constant bias term in the prediction of the future outputs. In addition, because of the use of step response models, the traditional techniques are not applicable to integrating systems, which are common in chemical process industries.

Lately, there have been efforts to interpret Model Predictive Control in a state-space framework. This not only permits the use of well-known state-space theorems, but also allows straightforward generalization to more complex cases such as systems with general stochastic disturbances and measurement noise. Li *et al.* (1989) and Navartil *et al.* (1988) showed that the step response model can be put into the general state-space model structure and presented an MPC technique using the tools available from stochastic optimal control theory. They showed how open-loop and closed-loop observers can be incorporated into the predictive control framework to improve regulatory control of MPC. Ricker (1990) showed how an MPC algorithm similar to the conventional MPC techniques can be developed based on a *general* state-space model. Lee *et al.* (1992a) proposed a state-space MPC technique applicable to general multi-rate sampled-data systems. In their work, offset-free control for nonstationary disturbances is assured by using a velocity form of the state-space model in which the states and outputs are expressed in terms of the *changes* in inputs and disturbances. Recently, Bitmead *et al.* (1990) presented a lucid and detailed analysis of the basic features inherent in all MPC algorithms from the viewpoint of Linear Quadratic Regulator and Linear Quadratic Gaussian Control theory.

Some reseachers, especially those active in the adaptive control area, have preferred ARMA or CARIMA type models over state-space models in developing MPC algorithms. Clarke *et al.* (1987a-b) developed what is known as "Generalized Predictive Control (GPC)," based on the CARIMA input-output model and showed its connection to LQ optimal control.

Robustness to modelling errors and measurement noise was incorporated into the algorithm through user-specified "observer" polynomials as well as by weighting and constraining the future input moves (Clarke, 1991).

The step response model, despite the disadvantage of needing many more parameters than the conventional state-space or CARIMA model, has been preferred by many industrialists because it is intuitive, needs less *a priori* information to identify and provides a means to construct the prediction vector in a natural way. Since the general state-space model or the CARIMA model includes the step response model as a special case, extension of the traditional MPC (step response) techniques to accommodate general stochastic disturbances and measurement noise based on the above-mentioned developments is straightforward, at least in concept. However, direct application of the state estimation technique (Li *et al.*, 1989; Navratil *et al.*, 1988) to step response models adds significant numerical complexity such as the requirement to solve a Riccati equation of potentially very large order. The order is generally equal to the number of step response coefficients times the number of outputs. Only when the disturbance dynamics are much faster than the manipulated variable dynamics and a short prediction horizon is chosen, the order may be reduced significantly. In the GPC framework, the "observer" polynomial is specified directly rather than by solving a Riccati equation (Clarke, 1991). Although this approach has worked well for simple systems, the optimal choice for the "observer" polynomials for general multivariable systems is often not obvious. Another drawback of using the input-output model for MPC is that the generalization to the multivariable case, although conceptually staightforward, is complex and not robust numerically.

In this article, two new results are presented that should broaden the scope of application and improve the performance and robustness of the traditional MPC techniques with minor modifications. We achieve these improvements while preserving the main features of the traditional MPC techniques that contributed to their success in practical environments: the simplicity of the algorithm and the use of the step response model. First, we present a state space model expressed in terms of step response parameters for systems of stable and/or integrating dynamics. Second, we extend the conventional MPC techniques to handle general stochastic disturbances and white measurement noise in an optimal way using state estimation techniques. Contrary to Li's work, however, our approach does not require solving a large-order Riccati equation. Instead, it is shown that the optimal observer can be calculated by solving a Riccati equation of significantly lower dimension. More importantly, for the case of integrated or dounble-integrated white noise disturbances filtered through *general* first order dynamics, the optimal observer is conveniently parametrized through a real parameter vector whose dimension is the same as the number of outputs. Each element of the parameter vector lies in $(0, 1]$ and therefore can be adjusted on-line. The adjustable parameters of the state observer directly affect the speed of the closed-loop response as well as robustness to measurement noise and model uncertainty.

Finally, the new technique is put in perspective with many existing conventional techniques such as DMC, IMC and GPC. For integrated white noise disturbance at the output, the parameters play the same role as the time constants of the first-order robustness filter and observer polynomial used in IMC and GPC respectively. We also show that, for more complex disturbances, such simple equivalence among the techniques may not be established. Several examples demonstrate that the proposed MPC technique is applicable to a wider range of control problems and leads to enhanced performance without introducing further complexity.

Although constraints are not discussed explicitly, the results given in this paper are pertinent to the prediction part of the algorithm and therefore apply to the constrained case as well.

2 State-Space Model Using Step Response Parameters

In this section, we demonstrate how the step response data can be put in a standard state-space form for stable and integrating systems. We extend the conventional step response model to include integrating dynamics in a manner such that all the desirable features of the step response model (such as its intuitiveness and flexible structure for identification) are retained. The extended state-space model includes the step-response model presented by Li *et al.* (1989) as a special case.

2.1 Model Form

The model we adopt in this work is the following state-space model represented by step response coefficients:

$$Y(k) = MY(k-1) + S\Delta u(k-1) + T\Delta d(k-1) \tag{1}$$

$$y(k) = NY(k) \tag{2}$$

$$\hat{y}(k) = y(k) + v(k) \tag{3}$$

where

$$Y(k) = [y_0^T(k), y_1^T(k), \cdots, y_{n-2}^T(k), y_{n-1}^T(k), x_p^T(k), x_d^T(k)]^T \tag{4}$$

$$M = \left.\begin{bmatrix} 0 & I_{n_y} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & I_{n_y} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & I_{n_y} & 0 & \\ 0 & 0 & 0 & \cdots & 0 & I_{n_y} & I_{n_y} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & A_p & C_d \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A_d \end{bmatrix}\right\} (n+1)\cdot n_y + \dim\{x_d\}; \tag{5}$$

$$N = [\,\overbrace{I_{n_y} \;\; 0 \;\; 0 \;\; \cdots \;\; 0 \;\; 0 \;\; 0}^{(n+1)\cdot n_y + \dim\{x_d\}}\,] \tag{6}$$

$$
S = \begin{bmatrix} S_1 \\ S_2 \\ \vdots \\ S_{n-1} \\ S_n \\ S_{n+1} - S_n \\ 0 \end{bmatrix}; \quad T = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ B_d \end{bmatrix}; \tag{7}
$$

$$
S_i = \begin{bmatrix} s_{1,1,i} & s_{1,2,i} & \cdots & s_{1,n_u,i} \\ s_{2,1,i} & s_{2,2,i} & \cdots & s_{2,n_u,i} \\ \vdots & \vdots & \ddots & \vdots \\ s_{n_y,1,i} & s_{n_y,2,i} & \cdots & s_{n_y,n_u,i} \end{bmatrix} \quad i = 1, \cdots, n \tag{8}
$$

$$
\tag{9}
$$

$y(k)$, $u(k)$ and $d(k)$ are output, input and disturbance vectors respectively. $\Delta u(k)$ and $\Delta d(k)$ are the changes in u and d from the previous sampling time. The vector $Y(k)$ represents dynamics states of the system and $\hat{y}(k)$ is the noise-corrupt measurement of $y(k)$. $s_{\ell,m,i}$ is the i^{th} step response coefficient relating the m^{th} input to the ℓ^{th} output. n_u and n_y are the number of inputs and outputs, respectively. A_p is a diagonal matrix of the following form:

$$
A_p = \begin{bmatrix} a_1 & & \\ & \ddots & \\ & & a_{n_y} \end{bmatrix} \tag{10}
$$

$$
a_i = \begin{cases} 0 & \text{if } i_{\text{th}} \text{ output is a stable (nonintegrating) output.} \\ 1 & \text{if } i_{\text{th}} \text{ output is an integrating output.} \end{cases} \tag{11}
$$

It is assumed here, after n time steps, all the effects of stable dynamics settle and the step responses of nonintegrating and integrating outputs remain constant and increase with a constant slope respectively. It is assumed that all the eigenvalues of A_d lie strictly inside the unit disk making the disturbance dynamics stable (except for the integrating dynamics already present in M).

Remarks:

1. *Dynamics States*
   Each element of the state vector $Y(k)$, $y_\ell(k)$, has a special physical interpretation: it is the output y at time $k + \ell$ assuming the input and disturbance remain constant starting at time $k - 1$ (or $\Delta u(k + \ell) = \Delta d(k + \ell) = 0$ for $\ell \geq 0$).

2. *Process Dynamics*
   In general, when the step response is truncated after n time steps, the triple $(A_p, S_{n+1} - S_n, I_{n_y})$ can be used to express the residual step response (see Hovd *et al* (1991), for example), but this hybrid step response / state-space model dilutes the main attractiveness of the step response model and is not discussed here.

3. *External Disturbance and Measurement Noise*
For state estimation, $\Delta d(k)$ is chosen as white noise with the following covariance matrix:

$$E\{\Delta d(k)\Delta d^T(k)\} = W = \begin{bmatrix} W_1 & & \\ & \ddots & \\ & & W_{\dim\{d\}} \end{bmatrix} \tag{12}$$

Without loss of generality, we assumed here a diagonal covariance matrix. This formulation makes the disturbance observed at the output integrated white noise filtered through the system $(qI - A_p)^{-1}C_d(qI - A_d)^{-1}B_d$ plus n delays (q denotes the forward-shift operator). n delays do not have any effect on the disturbance estimation since the signal resulting from passing white noise through n delays is white noise of the same intensity. For a stable output ($a_i = 0$), the disturbance observed at the output is simply an integrated white noise (*i.e.*, random steps) filtered through the stable dynamics of $C_d(qI - A_d)^{-1}B_d$. For an integrating output ($a_i = 1$), the disturbance dynamics contain an extra integrator which makes the disturbance effect observed at the output *double-integrated* white noise (*i.e.*, random ramps) filtered through $C_d(qI - A_d)^{-1}B_d$. This formulation is motivated by the fact that it is necessary to include double integrators in the disturbance dynamics for integrating outputs in order to design the optimal state estimator giving bias-free estimates. Measurement noise $v(k)$ is also white noise of covariance V.

One critical point to note is that the disturbance enters the future output state $y_{n-1}(k)$ first and is propagated down to the current state $y_0(k)$. It is also possible to add the disturbance effect directly to $y_0(k)$. However, in this case, the state $y_\ell(k)$ would lose the special physical interpretation of being the *future output* at time $k + \ell$ assuming the input and disturbance remain constant ($\Delta d(k + \ell) = \Delta u(k + \ell) = 0 \quad \forall \ell \geq 0$). Retaining this physical interpretation proves to be very useful in constructing the optimal prediction vector later on.

Some researchers (Navratil, 1988) have suggested the use of step response coefficients to describe the effect of disturbances on the output. In this case, T should contain the step response coefficients describing the change of the output caused by the changes in the disturbances. For unmeasured disturbances, however, it is often difficult to obtain good estimates for these step response coefficients. In addition, the calculation of the optimal state observer for an arbitrary T involves solving a Riccati equation of very high order. In the next section, we will show that, because of the special structure of the step response model and T chosen here, it is only necessary to solve a Riccati equation of dimension $\dim\{x_d\}$.

2.2 Stabilizability and Detectability

It is important to elucidate the requirements on M, N, S and T with respect to stabilizability and detectability to give rise to a meaningful problem definition. It should be clear that, for an investigation of these properties, we can drop all the "delays" from M and study the smaller system, M', N', S' and T' where

$$M' = \begin{bmatrix} I_{n_y} & I_{n_y} & 0 \\ 0 & A_p & C_d \\ 0 & 0 & A_d \end{bmatrix}; N' = \begin{bmatrix} I_{n_y} & 0 & 0 \end{bmatrix}; S' = \begin{bmatrix} S_n \\ S_{n+1} - S_n \\ 0 \end{bmatrix}; T' = \begin{bmatrix} 0 \\ 0 \\ B_d \end{bmatrix} \tag{13}$$

The requirements for the traditional LQG design are:

(a) $\{N', M'\}$ is a detectable pair.
(b) $\{M', S'\}$ is a stabilizable pair.
(c) $\{M', T'\}$ is a stabilizable pair.

When conditions (a) and (b) are satisfied, the Riccati difference equations for the infinite horizon LQ regulator and Kalman filter converge to unique, positive semi-definite solutions respectively (Goodwin & Sin, 1984). The other condition (c) guarantees that the corresponding LQG controller places all the closed-loop poles inside the unit disk. We will examine the above requirements and their implication in more detail. It is convenient to treat separately the case when A_p contains integrators.

Case 1: A_p is stable ($A_p = 0$).

(a) trivially satisfied.
(b) S_n must have full row rank.
(c) $C_d(I - A_d)^{-1}B_d$ must have full row rank.

Condition (b) requires $n_u \geq n_y$: In steady state, it must be possible to affect all the outputs y independently from the manipulated variables u. Condition (c) requires $n_d \geq n_y$ where n_d is the number of disturbances generated by the triple $\{C_d, A_d, B_d\}$. The disturbance model should be chosen such that, in steady state, it must be possible to affect all the outputs y independently from the disturbances d. This condition is necessary to ensure that the Kalman filter gain resulting from the steady-state solution to the Riccati difference equation places all the observer poles *inside* the unit disk. If this condition fails, the Kalman filter places some of the observer poles at $(1, 0)$. This causes the Kalman filter to give biased estimates in practice. While the infinite horizon Riccati equation for the LQ regulator design fails to converge if condition (b) is not satisfied, it is still possible to design a finite horizon predictive controller although such a controller would leave a steady-state offset in general.

Case 2: A_p contains integrators.

(a) trivially satisfied.
(b) $\begin{bmatrix} I_{n_y} & S_n \\ A_p - I & S_{n+1} - S_n \end{bmatrix}$ must have full row rank.
(c) $\begin{bmatrix} I_{n_y} & 0 \\ A_p - I & C_d(I - A_d)^{-1}B_d \end{bmatrix}$ must have full row rank.

The above conditions are straightforward generalization of the conditions given for Case 1 and each condition has the same interpretation and implication as before.

2.3 Special Case: Decoupled Integrated White Noise Disturbance Filtered Through First Order Dynamics and White Measurement Noise

In this article, we will focus on a particular choice of disturbance dynamics (A_d, B_d, C_d) and the noise covariance matrix V, namely

$$A_d = \mathcal{A} \triangleq \text{diag}\{\alpha_1, \cdots, \alpha_{n_y}\}, \quad 0 \le \alpha_i < 1; \qquad B_d = I_{n_y}; \qquad C_d = I_{n_y}; \tag{14}$$

$$V = E\{v(k)v(k)^T\} = \begin{bmatrix} V_1 & & \\ & \ddots & \\ & & V_{n_y} \end{bmatrix} \tag{15}$$

Hence, the disturbance at the i^{th} output is integrated (or double-integrated for an integrating output) white noise filtered through first order dynamics $\frac{1}{q-\alpha_i}$. For stable outputs ($a_i = 0$), the choice of $\alpha_i = 0$ makes the disturbance at the i^{th} output integrated white noise ("type 1" disturbance). With the assumption of $\Delta u(k) = 0 \;\; \forall k \ge 0$ and $y(0) = 0$,

$$y(k+n+2) = \sum_{j=0}^{k} \Delta d(j) \tag{16}$$

As $\alpha_i \to 1$, the disturbance at the i^{th} output approaches *double*-integrated white noise ("type 2" disturbance). At the limit, with the assumption of $\Delta u(k) = 0 \;\; \forall k \ge 0$ and $y(0) = 0$,

$$y(k+n+2) = \sum_{m=0}^{k} \sum_{j=0}^{m} \Delta d(j) \tag{17}$$

For integrating outputs, the presence of an extra integrator gives $\alpha = 0$ and $\alpha \to 1$ the interpretation of double-integrated and triple-integrated white noise disturbances at the output respectively. The disturbances at each output are assumed uncorrelated (by requiring that W be a diagonal matrix). The measurement noise at each output is also assumed to be uncorrelated, white noise.

Although the disturbance description (14) is admittedly limited in its generality, persistent, overdamped disturbances prevalent in most chemical processes are adequately described by it. The noise description (15) should be adequate for most practical problems as well. Another reason for concentrating on this particular disturbance/noise description is that, as will be shown later , we can obtain an explicit parametrization for the optimal filter gain and adjust the disturbance and noise parameters conveniently (possibly on-line) to obtain desirable loopshapes for robust performance.

3 State Estimation

In this section, we develop the optimal state estimation technique for the step response model (1)-(3); in other words, we will show how to estimate in an optimal fashion the dynamic states

$Y(k)$ on the basis of the measurements.

3.1 Optimal Estimator Form

For the system (1)-(3), the optimal estimator (*i.e.*, Kalman filter) based on the measurements at time k is most conveniently expressed in the following two-step form:
Model Prediction:

$$Y(k|k-1) = MY(k-1|k-1) + S\Delta u(k-1) \tag{18}$$

Correction Based on Measurements:

$$Y(k|k) = Y(k|k-1) + K\{\hat{y}(k) - NY(k|k-1)\} \tag{19}$$

The notation $Y(\ell|m)$ represents the estimate of $Y(\ell)$ based on the measurements up to time m. K is the optimal filter gain that can be calculated from (Åström & Wittenmark, 1984)

$$K = \Sigma_s N^T (N\Sigma_s N^T + V)^{-1} \tag{20}$$

where the $(n \cdot n_y + \dim\{x_p\} + \dim\{x_d\}) \times (n \cdot n_y + \dim\{x_p\} + \dim\{x_d\})$ matrix Σ_s is the steady-state solution (*i.e.*, asymptotic solution as $k \to \infty$) of the following Riccati difference equation:

$$\Sigma(k) = M\Sigma(k-1)M^T - M\Sigma(k-1)N^T(N\Sigma(k-1)N^T + V)^{-1}N\Sigma(k-1)M^T + TWT^T \tag{21}$$

Using the fact that $y_\ell(k|k)$ represents the optimal estimate of $y(k+\ell)$ based on measurements up to time k and assuming $\Delta u(k+j) = \Delta d(k+j) = 0 \quad \forall j \geq 0$, one can construct the optimal filter gain K by solving a Riccati equation of much smaller dimension. More specifically, consider the following reduced-order system:

$$\begin{bmatrix} y_0(k) \\ x_p(k) \\ x_d(k) \end{bmatrix} = \tilde{M} \begin{bmatrix} y_0(k-1) \\ x_p(k-1) \\ x_d(k-1) \end{bmatrix} + \tilde{T}\Delta d(k-1) \tag{22}$$

$$\hat{y}(k) = \tilde{N} \begin{bmatrix} y(k) \\ x_p(k) \\ x_d(k) \end{bmatrix} + v(k) \tag{23}$$

where

$$\tilde{M} = \begin{bmatrix} I_{n_y} & I_{n_y} & 0 \\ 0 & I_{n_y} & C_d \\ 0 & 0 & A_d \end{bmatrix}; \quad \tilde{T} = \begin{bmatrix} 0 \\ 0 \\ B_d \end{bmatrix}; \quad \tilde{N} = \begin{bmatrix} I_{n_y} & 0 & 0 \end{bmatrix} \tag{24}$$

Note that we have not included the effect of the manipulated inputs in the model since they are exactly known inputs to the system and do not affect the Kalman filter gain calculation. Other than that, the optimal estimate $y_0(k|k)$ should be same as before since the only change is that the disturbances are now entering the output after 1 delay instead of n delays. Let $\begin{bmatrix} K_a \\ K_b \\ K_c \end{bmatrix}$ be the optimal filter gain for the above reduced-order system. In other words,

$$\begin{bmatrix} K_a \\ K_b \\ K_c \end{bmatrix} = \tilde{\Sigma}_s \tilde{N}^T (\tilde{N}\tilde{\Sigma}_s \tilde{N}^T + V)^{-1} \tag{25}$$

where $\tilde{\Sigma}_s$ is the steady-state solution to the following Riccati difference equation:

$$\tilde{\Sigma}(k) = \tilde{M}\tilde{\Sigma}(k-1)\tilde{M}^T - \tilde{M}\tilde{\Sigma}(k-1)\tilde{N}^T(\tilde{N}\tilde{\Sigma}(k-1)\tilde{N}^T + V)^{-1}\tilde{N}\tilde{\Sigma}(k-1)\tilde{M}^T + \tilde{T}W\tilde{T}^T \tag{26}$$

Then, for the particular choice of T in (7), the optimal filter gain K is of the following form:

$$K = \begin{bmatrix} I_{n_y} \\ I_{n_y} \\ I_{n_y} \\ I_{n_y} \\ \vdots \\ \vdots \\ I_{n_y} \\ 0 \\ 0 \end{bmatrix} K_a + \begin{bmatrix} 0 \\ I_{n_y} \\ I_{n_y} + A_p \\ \sum_{i=0}^{2} A_p^i \\ \vdots \\ \vdots \\ \sum_{i=0}^{n-2} A_p^i \\ A_p^{n-1} \\ 0 \end{bmatrix} K_b + \begin{bmatrix} 0 \\ 0 \\ C_d \\ (I_{n_y} + A_p)C_d + C_d A_d \\ \vdots \\ \vdots \\ \sum_{j=0}^{n-3}\sum_{i=0}^{j} A_p^{j-i} C_d A_d^i \\ \sum_{i=0}^{n-2} A_p^{n-2-i} C_d A_d^i \\ A_d^{n-1} \end{bmatrix} K_c \tag{27}$$

The above formula can be constructed from the expression for the optimal estimates $y_0(k|k)$, $x_p(k|k)$ and $x_d(k|k)$ of the reduced order system, and then projecting them into the future while holding $\Delta d(k+j)$ to be zero for $j \geq 0$. This provides the measurement correction formula for $y(k+\ell|k)$, which is equivalent to $y_\ell(k|k)$ under the assumption that $\Delta d(k+j) = \Delta u(k+j) = 0 \ \ \forall j \geq 0$.

3.2 Special Case: Decoupled Integrated White Noise Disturbance Filtered Through First Order Dynamics and White Measurement Noise

It can be shown that, when the disturbance observed at each output is independent, integrated white noise filtered through first order dynamics (as described in (14)), the Kalman filter gain is in the form of (27) with:

$$K_a = \begin{bmatrix} (f_a)_1 & & \\ & \ddots & \\ & & (f_a)_{n_y} \end{bmatrix}; \quad K_b = \begin{bmatrix} (f_b)_1 & & \\ & \ddots & \\ & & (f_b)_{n_y} \end{bmatrix}; \quad K_c = \begin{bmatrix} (f_c)_1 & & \\ & \ddots & \\ & & (f_c)_{n_y} \end{bmatrix} \tag{28}$$

For stable outputs, $(f_b)_i$ and $(f_c)_i$ are related to $(f_a)_i$ through following equations:

$$(f_b)_i = \frac{\alpha_i (f_a)_i^2}{1+\alpha_i - \alpha_i (f_a)_i}; \qquad (f_c)_i = \frac{\alpha_i^2 (f_a)_i^2}{1+\alpha_i - \alpha_i (f_a)_i}; \tag{29}$$

Hence, the Kalman filter contains the adjustable parameters, $(f_a)_i$, that are determined by the disturbance-to-noise ratio for the i^{th} output, W_i/V_i. For the limiting cases,

$$(f_a)_i \rightarrow 0 \quad \text{for} \quad W_i/V_i \rightarrow 0 \tag{30}$$
$$(f_a)_i \rightarrow 1 \quad \text{for} \quad W_i/V_i \rightarrow \infty \tag{31}$$

Therefore, $(f_a)_i$ must lie in $(0, 1]$. For most chemical processes where the disturbances are persistent and overdamped, the disturbance description of (14) is adequate. Hence, for most industrial problems, there is *no* need to solve the Riccati equation; instead, $(f_a)_i$ and α_i can be used as tuning parameters that can be adjusted on-line.

For integrating systems, the expressions relating $(f_b)_i$ and $(f_c)_i$ with $(f_a)_i$ are somewhat complicated:

$$\begin{aligned}
(f_b)_i &= \frac{\beta_i + \sqrt{\gamma_i}}{\lambda_i}; \qquad (f_c)_i = \frac{\alpha_i^2 (f_b)_i^2}{(f_a)_i + (1-\alpha_i^2)(1-(f_a)_i) + \alpha_i((f_a)_i + (f_b)_i)} && (32)\\
\lambda_i &= 2\alpha_i(1-(f_a)_i) && (33)\\
\beta_i &= \alpha_i(\alpha_i+2)(f_a)_i^2 + (1+\alpha_i)(1-3\alpha_i)(f_a)_i - 2(1-\alpha_i)(1+\alpha_i) &&\\
\gamma_i &= \alpha_i^4 (f_a)_i^4 - 2\alpha_i^2(3\alpha_i+1)(\alpha_i+1)(f_a)_i^3 + (1+\alpha_i)^2(1-10\alpha_i+13\alpha_i^2)(f_a)_i^2 && (34)\\
&- 4(1-3\alpha_i)(1-\alpha_i)(1+\alpha_i)^2(f_a)_i + 4(\alpha_i+1)^2(\alpha_i-1)^2 && (35)
\end{aligned}$$

Note that, in the limiting cases of $\alpha_i \rightarrow 0$ and/or $(f_a)_i \rightarrow 1$, we cannot use the above formula for f_b as is, but L'Hôpital's rule can be applied to obtain

$$\lim_{\alpha_i \rightarrow 0} (f_b)_i = \frac{(f_a)_i^2}{2-(f_a)_i}; \qquad \lim_{(f_a)_i \rightarrow 1} (f_b)_i = 1 + \alpha_i \tag{36}$$

4 Prediction

The predictive controller computes the best current and future control moves based on the prediction of future outputs. The dynamic states $[y_0(k)^T, \cdots, y_{n-1}(k)^T]^T$ represent the current and future outputs assuming all current and future inputs are zero (*i.e.*, $\Delta u(k+j) = \Delta d(k+j) = 0$ for $j \geq 0$). In the previous section, we demonstrated how to obtain optimal estimates for these states. The optimal prediction of future outputs can be obtained simply by adding the effect of future input moves to the optimal state estimates $y_\ell(k|k)$. Since the unmeasured disturbance $\Delta d(k)$ is zero-mean white noise, it is optimal to develop the prediction with $\Delta d(k+j) = 0 \quad \forall j \geq 0$ (Åström & Wittenmark, 1984). Hence, the future outputs can be expressed in terms of current and $(m-1)$ future inputs through the following equation:

$$\mathcal{Y}(k+1|k) = M_p Y(k|k) + S_p^m \Delta \mathcal{U}(k) \tag{37}$$

where

$$M_p = \left[\begin{array}{cc} I_{p\cdot n_y \times p\cdot n_y} & 0 \end{array}\right] M \tag{38}$$

$$S_p^m = \left[\begin{array}{ccccc} S_1 & 0 & \cdots & \cdots & 0 \\ S_2 & S_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ S_m & S_{m-1} & \cdots & \cdots & S_1 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ S_p & S_{p-1} & \cdots & \cdots & S_{p-m+1} \end{array}\right]; \quad \Delta\mathcal{U}(k) = \left[\begin{array}{c} \Delta u(k) \\ \Delta u(k+1) \\ \vdots \\ \vdots \\ \Delta u(k+m-1) \end{array}\right] \tag{39}$$

The notation $\mathcal{Y}(k+1|k)$ denotes the predicted future outputs up to time $k+p$ for constant inputs starting at time $k+m$, based on the measurements up to time k. We allow the flexibility of considering only a specified number of input moves (which may be smaller than the prediction horizon p). Note that the formula (37) is applicable only to the case where $p \leq n$. When it is desired to choose a prediction horizon larger than the number of step response coefficients, the prediction equation (37) can be modified in a straightforward manner.

5 Feedback Control

We adopt the following quadratic optimzation objective (used in QDMC (Garcia & Morshedi, 1984)):

$$\min_{\Delta\mathcal{U}(k)} \|\Gamma[Y(k+1|k) - \mathcal{R}(k+1)]\|^2 + \|\Lambda\Delta\mathcal{U}(k)\|^2 \tag{40}$$

$\mathcal{R}(k+1) = [r^T(k+1), \cdots, r^T(k+p)]^T$ is the future output reference vector. Γ and Λ are weighting matrices that are chosen to be diagonal for most cases. This optimization problem can be cast into the following least-squares problem:

$$\left[\begin{array}{c} \Gamma S_p^m \\ \Lambda \end{array}\right] \Delta\mathcal{U}(k) = \left[\begin{array}{c} \Gamma\left(\mathcal{R}(k+1) - M_p Y(k|k)\right) \\ 0 \end{array}\right] \tag{41}$$

The least-squares solution is

$$\Delta\mathcal{U}(k) = \{(S_p^m)^T\Gamma^T\Gamma S_p^m + \Lambda^T\Lambda\}^{-1}(S_p^m)^T\Gamma^T\Gamma\left(\mathcal{R}(k+1) - M_p Y(k|k)\right) \tag{42}$$

The curent control move is implemented:

$$\Delta u(k) = [\begin{array}{cccc} I_{n_u} & 0 & \cdots & 0 \end{array}]\Delta\mathcal{U}(k) \tag{43}$$

The controller can be interpreted as a state-observer-based compensator since

$$\Delta u(k) = K_{MPC}(\mathcal{R}(k+1) - M_p Y(k|k)) \tag{44}$$

where

$$K_{MPC} = [\begin{array}{cccc} I_{n_u} & 0 & \cdots & 0 \end{array}]\{(S_p^m)^T\Gamma^T\Gamma S_p^m + \Lambda^T\Lambda\}^{-1}(S_p^m)^T\Gamma^T\Gamma \tag{45}$$

6 Closed-Loop Relationships

We can derive the closed-loop relationships between the actual process output $y(k)$ and the system inputs $w(k)$, $v(k)$ and $\mathcal{R}(k)$ using the following relationships:

$$\begin{aligned} Y(k) &= MY(k-1) + S\Delta u(k-1) + T\Delta d(k-1) && (46)\\ Y(k|k) &= (M-KNM)Y(k-1|k-1) + K\hat{y}(k) + (I-KN)S\Delta u(k-1) && (47)\\ \hat{y}(k) &= NMY(k-1) + NS\Delta u(k-1) + NT\Delta d(k-1) + v(k) && (48)\\ \Delta u(k-1) &= -K_{MPC}M_pY(k-1|k-1) + K_{MPC}\mathcal{R}(k) && (49) \end{aligned}$$

Simple algebraic manipulations lead to

$$\begin{aligned} \begin{bmatrix} Y(k) \\ Y(k|k) \end{bmatrix} &= \begin{bmatrix} M & -SK_{MPC}M_p \\ KNM & M-KNM-SK_{MPC}M_p \end{bmatrix} \begin{bmatrix} Y(k-1) \\ Y(k-1|k-1) \end{bmatrix} \\ &+ \begin{bmatrix} T & 0 & SK_{MPC} \\ KNT & K & SK_{MPC} \end{bmatrix} \begin{bmatrix} \Delta d(k-1) \\ v(k) \\ \mathcal{R}(k) \end{bmatrix} \end{aligned} \tag{50}$$

Subtracting the second equation from the first one, we obtain

$$\begin{aligned} \begin{bmatrix} Y(k) \\ Y(k) - Y(k|k) \end{bmatrix} &= \begin{bmatrix} M-SK_{MPC}M_p & SK_{MPC}M_p \\ 0 & M-KNM \end{bmatrix} \begin{bmatrix} Y(k-1) \\ Y(k-1) - Y(k-1|k-1) \end{bmatrix} \\ &+ \begin{bmatrix} T & 0 & SK_{MPC} \\ T-KNT & -K & 0 \end{bmatrix} \begin{bmatrix} \Delta d(k-1) \\ v(k) \\ \mathcal{R}(k) \end{bmatrix} \end{aligned} \tag{51}$$

The closed-loop operator (expressed in terms of pulse transfer function) from $\begin{bmatrix} \Delta d^T(k) & v^T(k) & \mathcal{R}^T(k) \end{bmatrix}^T$ to $y(k)$ can be written as follows:

$$y(k) = \left[\begin{array}{cc|ccc} M-SK_{MPC}M_p & SK_{MPC}M_p & T & 0 & SK_{MPC} \\ 0 & M-KNM & T-KNT & -K & 0 \\ \hline N & 0 & 0 & 0 & 0 \end{array}\right] \begin{bmatrix} \Delta d(k) \\ qv(k) \\ q\mathcal{R}(k) \end{bmatrix} \tag{52}$$

where

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] \triangleq C(qI-A)^{-1}B + D \tag{53}$$

Remarks:

1. *Closed-Loop Stability*
   The eigenvalues of the closed-loop matrix are those of $M-KNM$ and $M-SK_{MPC}M_p$. Hence, the closed-loop system is stable if and only if all eigenvalues of $M-KNM$ (*i.e.*, observer poles) and $M-SK_{MPC}M_p$ (*i.e.*, regulator poles) lie strictly inside the unit disk.

- The observer poles are guaranteed to lie inside the unit disk from the property of the Riccati equation.
- The regulator poles are functions of the tuning parameters (*e.g.*, p, m, Γ, Λ) and can be made to be stable by proper tuning.
- Under infinite input/output prediction horizon ($m = p = \infty$), the MPC regulator is equivalent to the LQ optimal regulator (computed from the steady-state solution of the Riccati equation) and places all the regulator poles inside the unit disk assuming the system is stabilizable and $\Gamma, \Lambda > 0$.
- The particular choice of $p = \infty$ and $m < \infty$ also guarantees to place all the regulator poles inside the unit disk under the same assumption (Rawlings & Muske, 1991). It can be reformulated as a finite horizon problem with $p = m$ by adding a terminal state weighting term. The major advantage of this approach over the infinite horizon LQ regulator is that regulator stability is retained even when constraints are entered into the algorithm. The terminal state weight is found by solving a Lyapunov equation.

2. *Tuning for Sensitivity and Robustness*
   The closed-loop expressions provide insights and guidelines for selecting various tuning parameters so that a desirable closed-loop response may be achieved.

   - Note that the observer dynamics affect the closed-loop transfer function from disturbance (Δd) and measurement noise (v), but not from the output reference vector ($\mathcal{R}$). On the other hand, the regulator dynamics affect all closed-loop tranfer functions. $\mathcal{R}$ may be filtered separately and this results in the classical two degrees-of-freedom controller (Morari & Zafiriou, 1989).
   - The closed-loop transfer function from $v(k)$ to $y(k)$ is the complementary sensitivity function which has a direct relevance to the closed-loop system's sensitivity and robustness. Observer poles, which are adjusted through the filter parameters, directly affect the complementary sensitivity function. Hence, the adjustable parameters we introduced for the estimator can be used to adjust the speed of disturbance response and closed-loop robustness.

3. *Asymptotic Disturbance Rejection Property*
   The closed-loop system rejects "persistent" disturbances with no offset as long as the observer/regulator poles are placed inside the unit disk. This can be seen from the closed-loop relationship from $\Delta d(k)$ to $y(k)$: $y(k)$ is simply expressed as a white noise filtered through stable (closed-loop) dynamics and therefore has a finite variance.

7 Connection with Other MPC Techniques

In this section, we make a connection between the new state-estimation-based MPC technique and other MPC techniques such as Dynamic Matrix Control (Cutler & Ramaker, 1980), Internal Model Control (Garcia & Morari, 1982) and Generalized Predictive Control (Clarke *et al.*, 1987a-b). The discussion in this section is limited to open-loop stable systems.

7.1 Connection with DMC and IMC

In most traditional MPC techniques including DMC and IMC, the following open-loop observer is used:

Model Update

$$\tilde{Y}(k) = M\tilde{Y}(k-1) + S\Delta u(k-1) \tag{54}$$

While the model states keep track of the effect of manipulated variable moves on the future outputs, the effect of unmeasured disturbances is unaccounted for. Hence,the effect of disturbances must be included in the prediction of the future outputs. The following equation based on the open-loop observer (54) gives a prediction which is equal to that resulting from the optimal closed-loop observer (18)-(19):

Prediction with Correction Based on Measurements

$$\mathcal{Y}(k+1|k) = M_p\tilde{Y}(k) + S_p^m \Delta\mathcal{U}(k) + \mathcal{T}^p(\hat{y}(k) - N\tilde{Y}(k)) \tag{55}$$

where

$$\mathcal{T}^p = M_p\left[I - q^{-1}(M - KNM)\right]^{-1} K \tag{56}$$

K is the optimal Kalman filter gain (20) or (28) for the specific case of disturbances in the form of integrated white noise filtered through first-order dynamics.

Connection with DMC

In DMC, the following equation is used for prediction:

$$\mathcal{Y}(k+1|k) = M_p\tilde{Y}(k) + S_p^m \Delta\mathcal{U}(k) + \mathcal{T}^p_{DMC}(\hat{y}(k) - N\tilde{Y}(k)) \tag{57}$$

where

$$\mathcal{T}^p_{DMC} \triangleq \left[\overbrace{\begin{matrix} I_{n_y} & \cdots & \cdots & I_{n_y} \end{matrix}}^{p}\right]^T \tag{58}$$

If we assume that the disturbance at each output is integrated white noise and measurements are noise-free and substitute the appropriate filter gain ((28) with $\alpha_i = 0, (f_a)_i = 0 \quad \forall i$) into (56), we obtain that $\mathcal{T}^p = \mathcal{T}^p_{DMC}$. Hence, DMC constructs the prediction vector assuming an independent, integrated white noise disturbance at each output and noise-free measurements. We have shown here that the standard DMC algorithm can be modified in a straightforward manner for optimal prediction (*e.g.*, (56)) in the presence of more general disturbances and noisy measurements.

Connection with IMC

In Internal Model Control (IMC) (see Figure 1), $\mathcal{T}^p$ is separated into two terms as follows:

$$\mathcal{T}^p = \mathcal{T}^p_{\text{nf}} \cdot F \tag{59}$$

where $\mathcal{T}^p_{\text{nf}} = M_p \{I - q^{-1}(M - KNM)\}^{-1} K_{\text{nf}}$. Here K_{nf} is the optimal filter gain for the noise-free case, and F is a low-pass filter matrix. The IMC design is normally carried out using an input/output transfer function model by first constructing the optimal IMC controller Q_{IMC} through model inversion (see Figure 1) and then augmenting it with a low-pass filter F which detunes the loop for robustness and measurement noise attenuation (Morari & Zafiriou, 1989). We next show that, in some special cases, we can construct F that will result in the same prediction as that constructed from the optimal state estimation.

For integrated white noise disturbance, by substituting K of (28) with $\alpha_i = 0$ into (56), we obtain

$$\mathcal{T}_p = \begin{bmatrix} I_{n_y} \\ \vdots \\ \vdots \\ I_{n_y} \end{bmatrix} \begin{bmatrix} \frac{(f_a)_1 q}{q-(1-(f_a)_1)} & & \\ & \ddots & \\ & & \frac{(f_a)_{ny} q}{q-(1-(f_a)_{ny})} \end{bmatrix} \tag{60}$$

Since $\mathcal{T}^p_{\text{nf}} = \mathcal{T}^p_{DMC}$ in this case, the following filter leads to the prediction equivalent to that from equations (18)-(19):

$$F = \begin{bmatrix} \frac{(f_a)_1 q}{q-(1-(f_a)_1)} & & \\ & \ddots & \\ & & \frac{(f_a)_{ny} q}{q-(1-(f_a)_{ny})} \end{bmatrix} \tag{61}$$

For stable systems with the disturbance (14), $\mathcal{T}^p$ in the prediction equation (55) has the form

$$\mathcal{T}^p = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_p \end{bmatrix} \tag{62}$$

where

$$F_1 = \begin{bmatrix} \frac{\{(f_a)_1+(f_b)_1\}q^2-\alpha_1(f_a)_1 q}{q^2-\{\alpha_1+1-(f_a)_1-(f_b)_1\}q+\alpha_1\{1-(f_a)_1\}} & & \\ & \ddots & \\ & & \frac{\{(f_a)_{ny}+(f_b)_{ny}\}q^2-\alpha_{ny}(f_a)_{ny} q}{q^2-\{\alpha_{ny}+1-(f_a)_{ny}-(f_b)_{ny}\}q+\alpha_{ny}\{1-(f_a)_{ny}\}} \end{bmatrix} \tag{63}$$

$$F_2 = \begin{bmatrix} \frac{\{(f_a)_1+(1+\alpha_1)(f_b)_1\}q^2-\alpha_1\{(f_a)_1+(f_b)_1\}q}{q^2-\{\alpha_1+1-(f_a)_1-(f_b)_1\}q+\alpha_1\{1-(f_a)_1\}} & & \\ & \ddots & \\ & & \frac{\{(f_a)_{ny}+\{1+\alpha_{ny}\}(f_b)_{ny}\}q^2-\alpha_{ny}\{(f_a)_{ny}+(f_b)_{ny}\}q}{q^2-\{\alpha_{ny}+1-(f_a)_{ny}-(f_b)_{ny}\}q+\alpha_{ny}\{1-(f_a)_{ny}\}} \end{bmatrix} \tag{64}$$

$$\vdots$$

The general formula for F_i is as follows:

$$F_i = \begin{bmatrix} (F_i)_1 & & \\ & \ddots & \\ & & (F_i)_{n_y} \end{bmatrix} \qquad 2 \le i \le p \tag{65}$$

where

$$(F_i)_j = \frac{\{(f_a)_j + (f_b)_j \sum_{k=1}^{i} \alpha_j^{k-1}\}q^2 - \alpha_1\{(f_a)_j + (f_b)_j \sum_{k=1}^{i-1} \alpha_j^{k-1}\}q}{q^2 - \{\alpha_j + 1 - (f_a)_j - (f_b)_j\}q + \alpha_j\{1 - (f_a)_j\}} \qquad 1 \le j \le n_y \tag{66}$$

Except for the cases where $\alpha_i = 0, \forall i$ and/or $p = 1$, there is *no* F such that

$$\mathcal{T}^p = \mathcal{T}_{\text{nf}}^p \cdot F \tag{67}$$

This result implies that, for systems with the more general disturbance (14), there exists no IMC filter that will give the same prediction as the optimal state estimator. Hence, the IMC filter leading to the same performance as the optimal-state-estimation-based MPC will be generally quite complex involving the state feedback parameters as well as those of the state estimator. An exception is the case when the prediction horizon is chosen to be same as the number of delays from the manipulated variable to the output. In this case, choosing the IMC filter

$$F = \left\{ \left[\overbrace{\begin{matrix} 0 & \cdots & 0 \end{matrix} \; I_{n_y}}^{p \cdot n_y} \right] \mathcal{T}_{\text{nf}}^p \right\}^{-1} F_p \tag{68}$$

yields an IMC controller equivalent to the state-estimation-based MPC since the prediction up to time $k + p - 1$ doesn't affect the control move because of the p-unit delay.

Even though traditional algorithms using the open-loop observer (54) can be modified for optimal prediction as just shown, for integrating systems, the open-loop observer leads to an "internally unstable" closed-loop system (the signal $\hat{y}(k) - N\tilde{Y}(k)$ can grow unbounded). This internal instability arises from the fact that $N\tilde{Y}(k)$ is not an estimate of the true output since it does not account for the effect of the disturbances. The approach discussed in this paper was based on a closed-loop observer and does not suffer from the same deficiency.

7.2 Connection with GPC

In GPC (Clarke *et al.*, 1987a-b; Clarke, 1991), the following CARIMA model is used:

$$A(q^{-1})y(k) = B(q^{-1})u(k) + \frac{C(q^{-1})T(q^{-1})}{1 - q^{-1}} \Delta d(k) \tag{69}$$

where $A(q^{-1}), B(q^{-1}), C(q^{-1})$ and $T(q^{-1})$ are polynomials of the backward-shift operator q^{-1}. We concentrate here on the single-input/single-ouput case for simplicity. Robustness is achieved by specifying the observer polynomial $T(q^{-1})$. In order to obtain a disturbance model equivalent to (14), the polynomials A and C should be chosen as follows:

$$A(q^{-1}) = (1-\alpha q^{-1})\tilde{A}(q^{-1}) \tag{70}$$

$$C(q^{-1}) = \tilde{A}(q^{-1}) \tag{71}$$

where $\tilde{A}(q^{-1})$ is the polynomial expressing the dynamics from the manipulated variable to the output.

For $\alpha = 0$ (*i.e.*, integrated white noise disturbance), simple calculation shows that choosing $T(q^{-1}) = 1-(1-f_a)q^{-1}$ results in the same prediction vector as (37). This is indeed the observer polynomial that Clarke (1991) recommends. For general α, $T(q^{-1}) = 1-(\alpha+1-f_a-f_b)q^{-1}+\alpha(1-f_a)q^{-2}$ leads to the equivalent prediction as (37). While such an equivalence can be established, the choice of the optimal observer polynomial would not have been obvious.

7.3 IMC - LQG/LTR Tuning Strategies

The IMC design philosophy (Morari & Zafiriou, 1989) is to make the IMC controller (Q in Figure 1) to be close to the inverse of the plant model ($\tilde{P}^{-1}$). This assures that the complementary sensitivity function can be freely chosen by the user-specified parameter F. In our framework, we can take a similar approach: Namely, we may use K_{MPC} obtained by letting the input weight approach zero (*i.e.*, $\Lambda \to 0$) and use the filter parameters (having a direct connection to F) as the only adjustable parameters to achieve robustness. This approach is in the same spirit as LQG/LTR (Doyle,1981; Bitmead *et al.*, 1991) in which the robustness margin of the Kalman filter (or LQ regulator) is recovered by the same procedure.

For stable, minimum-phase systems with integrated white noise disturbances, it can be shown that the closed-loop transfer function from the output disturbance to the output (frequently referred to as "sensitivity function") for $\Lambda = 0$ is as follows:

$$\mathcal{F}_{yd} = I - \begin{bmatrix} \frac{(f_a)_1}{q-(1-(f_a)_1)} & & \\ & \ddots & \\ & & \frac{(f_a)_{ny}}{q-(1-(f_a)_{ny})} \end{bmatrix} \tag{72}$$

Hence, for minimum-phase systems, the state-estimation-based MPC with zero input weighting gives a first-order closed-loop response of time constant $-T/\ln(1-(f_a)_i)$. For stable, minimum-phase systems with the disturbance (14), the closed-loop transfer function from the output disturbance to the output for $\Lambda = 0$ is as follows:

$$\mathcal{F}_{yd} = I - \begin{bmatrix} \frac{\{(f_a)_1+(f_b)_1\}q-\alpha_1(f_a)_1}{q^2-\{\alpha_1+1-(f_a)_1-(f_b)_1\}q+\alpha_1\{1-(f_a)_1\}} & & \\ & \ddots & \\ & & \frac{\{(f_a)_{ny}+(f_b)_{ny}\}q^2-\alpha_{ny}(f_a)_{ny}q}{q^2-\{\alpha_{ny}+1-(f_a)_{ny}-(f_b)_{ny}\}q+\alpha_{ny}\{1-(f_a)_{ny}\}} \end{bmatrix} \tag{73}$$

Numerical experience suggests that adjusting f_a leads to the loop-shapes desirable from robust control standpoint (Morari & Zafiriou, 1989; Lee & Yu, 1992b). Namely, the shape of the "non-detuned" sensitivity function (determined by the choice of α) is retained in the low frequency region and the complementary sensitivity function is rolled off starting at a frequency determined by the choice of f_a.

This IMC tuning approach simplifies controller tuning considerably as f_a detunes the loop in a specific manner regardless of the process dynamics. In addition to f_a, α can be adjusted to on-line in the case that the time constant of the disturbance dynamics is unknown or changes frequently. However, for "ill-conditioned" MIMO systems such as a high-purity distillation column, the input weighting may serve as a useful tuning parameter since it can prevent the control system from being "directionally sensitive" (Lee *et al.*, 1992a).

8 Numerical Example

8.1 Example A: Distillation Column Base Level Control

Problem Description

The behavior of the liquid level in the column base of a distillation column can be described as follows:

$$y(s) = Pu(s) + d(s) \tag{74}$$

where $u(s)$ represents the steam input (manipulated variable) and $d(s)$ represents the effect of various disturbances on the liquid level. The following model form was found to describe the behavior of many industrial columns adequately (Buckley *et al.*, 1985; Morari & Zafiriou, 1989):

$$P = \frac{1}{s}(1 - 2e^{-\theta s}) \tag{75}$$

Hence, it is an integrating system and exhibits inverse response behavior. The objective of the closed-loop control is to maintain a constant liquid level in the face of disturbances d. In this example, we treat the following two types of disturbances:

$$d(s) = d_I \triangleq \frac{1}{s}P(s) \quad \text{Step Disturbance at the Input} \tag{76}$$

$$d(s) = d_O \triangleq \frac{1}{s^2} \quad \text{Ramp Disturbance at the Output} \tag{77}$$

In practice, the dead-time θ is often not known exactly. To investigate the robustness of MPC controllers to dead-time uncertainty, we choose the following transfer functions as the model and the real plant:

<u>*Model*</u>

$$\tilde{P} = \frac{1}{s}(1 - 2e^{-s}) \tag{78}$$

Plant

$$P = P_0 \triangleq \frac{1}{s}(1 - 2e^{-s}) \tag{79}$$
$$P = P_- \triangleq \frac{1}{s}(1 - 2e^{-0.5s}) \tag{80}$$
$$P = P_+ \triangleq \frac{1}{s}(1 - 2e^{-1.5s}) \tag{81}$$
$$\tag{82}$$

When the plant is described by P_0, the model matches the plant exactly. When the plant is described either by P_- or by P_+, the model has a dead-time error of 1/2 minute.

Results from State-Estimation-Based MPC

Since the system is an integrating system with pure ramp disturbances, weapplied the state-estimation technique described in Section 3 $A_p = 1$ and $\alpha = 0$. The following MPC parameters were used:

- Sampling Time: 0.1 minute
- Number of Step-Response Coefficients (n): 50
- Prediction Horizon (p): 50 sampling units
- Number of Calculated Input Moves (m): 10 sampling units
- Input Weighting: 0
- Output Weighting: 1

The filter parameter f_a was varied to examine its effect on the robustness of the resulting closed-loop system. The closed-loop responses to the disturbances d_I and d_O (starting at t=1) for $P = P_0, P_-$, and P_+ are shown in Figures 2 - 4 respectively. In order to stabilize the closed-loop system with 1/2 minute delay errors ($P = P_-$ or $P = P_+$), the parameter f_a had to be chosen as low as 0.1 (choosing $f_a = 0.2$ resulted in instability for $P = P_+$). The simulations show that the filter parameter f_a indeed determines the speed of the closed-loop response and can be used to affect the robustness of the closed-loop system.

8.2 Example B: SISO System with "Slow" Disturbances

Problem Description

Let us consider a single-input/single-output system described by

$$y(s) = \frac{100}{s+1}u(s) + d(s) \tag{83}$$

and subjected to the following disturbances:

$$d(s) \;=\; d_A(s) \triangleq \left(\frac{100}{100s+1}\right)\frac{1}{s} \quad \text{Disturbance A} \tag{84}$$

$$d(s) \;=\; d_B(s) \triangleq \frac{1}{s} \quad \text{Disturbance B} \tag{85}$$

Hence, Disturbance A is a step disturbance added to the output through "slow" dynamics and Disturbance B is simply a step disturbance added to the output directly.

Results

We use the state-estimation-based MPC to minimize the effect of the disturbances on the output. The sampling time, prediction horizon, number of input moves, and input/output weights are chosen as in Example A. We compare the results obtained from using two different types of state estimators: a Type 1 estimator ($\alpha = 0$) for which the disturbance is assumed to be integrated white noise and a Type 2 estimator ($\alpha \rightarrow 1$) for which the disturbance is assumed to be double-integrated white noise. Figure 5 shows the closed-loop simulations of the output to Disturbances A and B (starting at t=1) under the MPC controller with a Type 1 estimator. Figure 6 shows the same closed-loop simulations when the Type 1 estimator is replaced by a Type 2 estimator. Although the MPC controller with the Type 1 estimator rejects Disturbance B (a step disturbance at the output) adequately, the responses of the output to Disturbance A (a "slow" disturbance) with the same controller are poor. The settling times for all values of f_a are unacceptably long. This is because an MPC controller with a Type 1 estimator projects the future outputs assuming the disturbance remains constant in the future; this is clearly not justified for Disturbance A. On the other hand, for the MPC controller with a Type 2 estimator, the responses of the output to Disturbance A are completely adequate. This improvement is due to the fact that an MPC controller with a Type 2 estimator projects the future outputs assuming that the *slope* of the disturbance remains constant in the future. For disturbance A, this assumption is well justified for the chosen prediction horizon. While the responses of the output to Disturbance B are not as good as those obtained for the Type 1 estimator, they are also quite acceptable.

9 Conclusions

In this article, we presented a state-space formulation of Model Predictive Control. By extending the conventional step response model and using the state-estimation techniques, we showed that MPC can be generalized to integrating systems and systems with white measurement noise without introducing additional complexity to MPC. We showed that under simple, but meaningful disturbance/noise assumptions, the optimal estimator can be parametrized in terms of a real parameter vector that can be used for on-line tuning. The state-space perspective also led to very simple tuning rules for stability and robustness: namely, the MPC controller can be interpreted as a state-observer-based compensator and its stability, performance and robustness are determined by the observer poles (which can be determined directly from the introduced adjustable parameter) and the regulator poles (which are determined by prediction horizon, input weighting, *etc.*). Based on the explicit parametrization of the optimal filter gain, we were able to make a connection between the new technique and other MPC techniques such as Internal Model Control, Dynamic Matrix Control and Generalized Predictive Control. Several examples demonstrated that the new state-estimation-based MPC can treat a wider range of problems for which the traditional techniques either would not have been applicable or would have led to poor results regardless of tuning.

All the results presented in this paper can be generalized in a straightforward manner to general state-space models and can be found in Lee & Yu (1992b).

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Figure 1: Block Diagram of Open-Loop-Observer-Based IMC Algorithm

Figure 2: Responses of the Output to Input/Output Disturbances for $P = P_0$ under State-Estimation-Based MPC

Figure 3: Responses of the Output to Input/Output Disturbances for $P = P_-$ under State-Estimation-Based MPC

Figure 4: Responses of the Output to Input/Output Disturbances for $P = P_+$ under State-Estimation-Based MPC

Figure 5: Responses of the Output to Disturbances A and B under MPC with a Type 1 Estimator

Figure 6: Responses of the Output to Disturbances A and B under MPC with a Type 2 Estimator

Figures not available via FTP